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EXACT SOLUTION FOR VIBRATION ANALYSIS OF RECTANGULAR CABLE NETWORKS WITH PERIODICALLY DISTRIBUTED SUPPORTS

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Rectangular cable networks supported by periodically distributed posts can be regarded as a structure with bi-periodicity in two orthogonal directions. By considering an equivalent system with cyclic bi-periodicity and applying the double U-transformation technique twice, the harmonic vibration equation can be uncoupled into a set of single variable equation, and that leads to the exact solutions. As an example, a square cable network with a 6×6 mesh and 2×2 internal supports is considered. The solutions of natural and forced vibrations are worked out by using the formulas obtained in the present study.

1. INTRODUCTION

The static and dynamic analysis of cable networks or beam grillages has been conducted by using many different methods, such as calculus of finite difference [1, 2], the membrane analogy method [3], the transfer matrix method [4], and the U-transformation method [5, 6]. Recently the U-transformation method has been applied to the analysis of mode localization phenomena in disordered cable networks by the authors [7].

Networks having periodic supports fall into the category of bi-periodic structures. The earliest study on bi-periodic structures might have been the analysis of compound periodic structures by Lin and McDaniel [8] using the transfer matrix method. The wave propagation in bi-periodic structures was investigated by Gupta [9] and Mead [10, 11] using the wave approach. The dynamics of bi-periodic structures was studied by McDaniel and Carroll [12]. More recently, a continuous beam with equi-distance rigid and elastic supports, subjected to a concentrated load was analyzed by the authors [13] using the U-transformation technique [14] and the exact solution was derived.

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However, the authors are unaware of any exact method for the dynamic problem of cable networks with periodically distributed supports. In this study, the natural and forced vibration analyses of the cable networks having bi-periodicity in two orthogonal directions are investigated by using the U-transformation technique.

Applying the double U-transformation [5] two times altogether to the governing equations for natural or forced vibration, the equations can be uncoupled into a set of single degree of freedom equations, leading to the exact solution for natural frequency or dynamic displacement.

The present method can be applied to static and dynamic analysis of rectangular beam grids and diagonal networks with periodically distributed supports. It is believed that this is the first analytical exact solution for this kind of problem.

2. UNCOUPLING OF THE DYNAMIC EQUATION AND DERIVATION OF THE EXPLICIT SOLUTION

The network considered is made up of two sets of pretensioned straight cables orthogonal to each other with fixed ends, meeting at spot-welded nodes and supported by periodically distributed posts. For generality, consider an $n_1 \times n_2$ rectangular network with fixed ends at four edges as shown in Figure 1 where the solid circles denote the nodes supported by the posts. There are $(m_1 - 1) \times (m_2 - 1)$ internal supports.

The equivalent network with cyclic periodicity in x- and y-directions can be produced by using image method [5, 6]. At the outset, consider the extended network with $2n_1 \times 2n_2$ mesh shown in Figure 2 where the loading pattern is anti-symmetric with respect to two symmetric planes of the extended network. Moreover we regard the extended network as one having cyclic bi-periodicity in x- and y-directions, i.e., each pair of nodes $(0, k)$ and $(2n_1, k)$ $(k = 0, 1, 2, \ldots, 2n_2)$

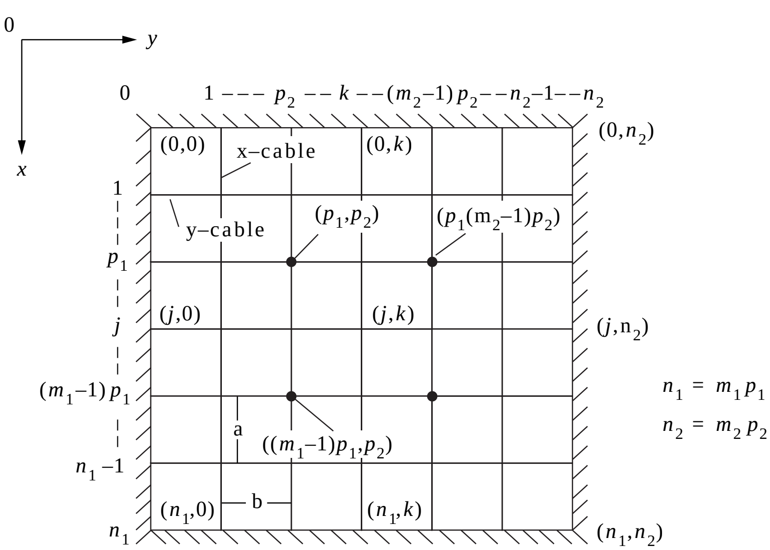

Figure 1. $n_1 \times n_2$ network with $(m_1 - 1) \times (m_2 - 1)$ supports.

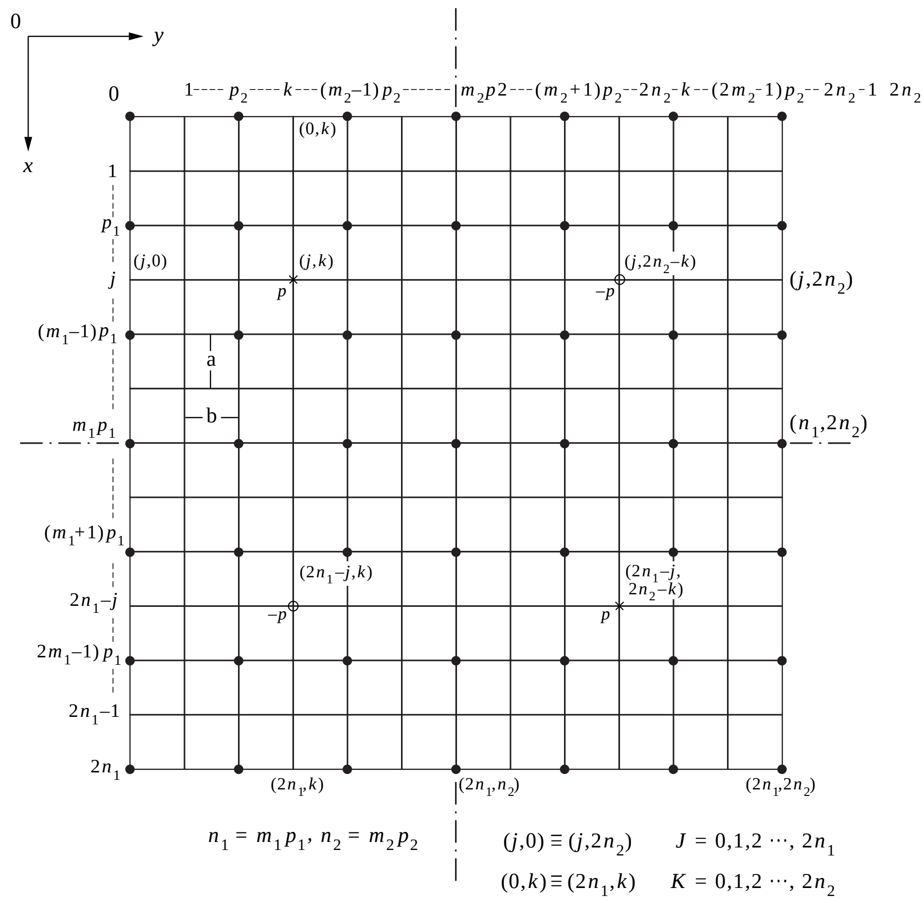

Figure 2. Equivalent network with $2n_1 \times 2n_2$ mesh and cyclic periodicity in x- and y-directions.

and $(j, 0)$ and $(j, 2n_2)$ $(j = 0, 1, 2, \ldots, 2n_1)$ may be imaginarily put together and treated as one point in mathematics. The boundary condition of the original system can be satisfied automatically in its equivalent network where the additional supports located at boundary and symmetric lines are necessary in order to form the cyclic bi-periodic system, but their supporting reactions being identically equal to zero.

2.1. HARMONIC INFLUENCE COEFFICIENT

Consider the equivalent network with no supports and lumped mass M for each node. The harmonic vibration equation for all nodes can be expressed as

$$(2K_1 + 2K_2 - M\omega^2)w_{(j,k)} - K_1(w_{(j+1,k)} + w_{(j-1,k)}) - K_2(w_{(j,k+1)} + w_{(j,k-1)}) = F_{(j,k)}$$

$$j = 1, 2, \ldots, 2n_1, \qquad k = 1, 2, \ldots, 2n_2 \tag{2.1}$$

$$K_1 = \frac{T_1}{a}, \qquad K_2 = \frac{T_2}{b} \tag{2.2}$$

with $w_{(2n_1+1,k)} \equiv w_{(1,k)}$, $w_{(j,2n_2+1)} \equiv w_{(j,1)}$ due to the cyclic periodicity, in which $w_{(j,k)}$ and $F_{(j,k)}$ denote the amplitudes of the transverse displacement and loading of node (j,k) respectively; ω denotes the vibration frequency; T_1, T_2 denote the pretensions of the cables in the x- and y-directions and a, b denote the spacing of y- and x-cables respectively.

In order to uncouple the simultaneous equations (2.1), introduce the double U-transformation

$$w_{(j,k)} = \frac{1}{\sqrt{2n_1}\sqrt{2n_2}} \sum_{r=1}^{2n_1} \sum_{s=1}^{2n_2} e^{i(j-1)r\psi_1}\, e^{i(k-1)s\psi_2} q_{(r,s)}$$
$$j = 1, 2, \ldots, 2n_1; \qquad k = 1, 2, \ldots, 2n_2; \tag{2.3a}$$

with the inverse transformation

$$q_{(r,s)} = \frac{1}{\sqrt{2n_1}\sqrt{2n_2}} \sum_{j=1}^{2n_1} \sum_{k=1}^{2n_2} e^{-i(j-1)r\psi_1}\, e^{-i(k-1)s\psi_2} w_{(j,k)}$$
$$r = 1, 2, \ldots, 2n_1; \qquad s = 1, 2, \ldots, 2n_2; \tag{2.3b}$$

and

$$\psi_1 = \pi/n_1, \qquad \psi_2 = \pi/n_2, \qquad i = \sqrt{-1}. \tag{2.4}$$

The harmonic vibration equation (2.1) can be expressed in terms of the generalized displacement $q_{(r,s)}$ as

$$(2K_1 + 2K_2 - M\omega^2 - 2K_1 \cos r\psi_1 - 2K_2 \cos s\psi_2) q_{(r,s)} = f_{(r,s)}$$
$$r = 1, 2, \ldots, 2n_1; \qquad s = 1, 2, \ldots, 2n_2 \tag{2.5}$$

where

$$f_{(r,s)} = \frac{1}{\sqrt{2n_1}\sqrt{2n_2}} \sum_{j=1}^{2n_1} \sum_{k=1}^{2n_2} e^{-i(j-1)r\psi_1}\, e^{-i(k-1)s\psi_2} F_{(j,k)}. \tag{2.6}$$

For the equivalent network, the loading must be anti-symmetric with respect to two symmetric planes, i.e.,

$$F_{(2n_1-j,2n_2-k)} = F_{(j,k)}$$
$$F_{(2n_1-j,k)} = F_{(j,2n_2-k)} = -F_{(j,k)}, \qquad j = 1, 2, \ldots, n_1; \qquad k = 1, 2, \ldots, n_2. \tag{2.7}$$

Substituting equation (2.7) into equation (2.6) yields

$$f_{(r,s)} = \frac{-4\, e^{ir\psi_1}\, e^{is\psi_2}}{\sqrt{2n_1}\sqrt{2n_2}} \sum_{j=1}^{n_1-1} \sum_{k=1}^{n_2-1} \sin jr\psi_1 \sin ks\psi_2 F_{(j,k)}. \tag{2.8}$$

As a result

$$f_{(r,s)} \equiv 0 \qquad r = n_1, 2n_1 \qquad \text{or} \qquad s = n_2, 2n_2 \tag{2.9a}$$

and

$$q_{(r,s)} \equiv 0 \qquad r = n_1, 2n_1 \qquad \text{or} \qquad s = n_2, 2n_2. \tag{2.9b}$$

Consider now the natural vibration, i.e., $f_{(r,s)} = 0$. The independent frequency equation for the network without supports can be expressed as

$$2K_1 + 2K_2 - M\omega^2 - 2K_1 \cos r\psi_1 - 2K_2 \cos s\psi_2 = 0$$
$$r = 1, 2, \ldots, n_1 - 1; \qquad s = 1, 2, \ldots, n_2 - 1; \tag{2.10a}$$

or

$$\omega^2 = \frac{1}{M}(2K_1 + 2K_2 - 2K_1 \cos r\psi_1 - 2K_2 \cos s\psi_2). \tag{2.10b}$$

Now consider the natural mode. Noting that $\cos(2n_1 - r)\psi_1 \equiv \cos r\psi_1$ and $\cos(2n_2 - s)\psi_2 \equiv \cos s\psi_2$, when we replace r or/and s by $2n_1 - r$ or/and $2n_2 - s$ respectively, the frequency equation (2.10a) is invariable. Therefore there are four non-trivial solutions, $q_{(r,s)}$, $q_{(2n_1-r,s)}$, $q_{(r,2n_2-s)}$, $q_{(2n_1-r,2n_2-s)}$ of equation (2.5) with $f_{(r,s)}$ vanishing corresponding to each natural frequency. The four generalized displacements with non-zero values represent four rotating modes of the equivalent network with cyclic periodicity. These rotating modes move in the positive or negative directions of x- and y-axes respectively.

Because of the anti-symmetry of the displacement $w_{(j,k)}$, equations (2.3b) can be written as

$$q_{(r,s)} = \frac{-4\, e^{ir\psi_1} e^{is\psi_2}}{\sqrt{2n_1}\sqrt{2n_2}} \sum_{j=1}^{n_1-1} \sum_{k=1}^{n_2-1} \sin jr\psi_1 \sin ks\psi_2 w_{(j,k)} \tag{2.11}$$

which indicates that $q_{(r,s)}$ corresponding to the anti-symmetric mode must involve the complex factor $e^{ir\psi_1} e^{is\psi_2}$ and a real factor $\sum_{j=1}^{n_1-1} \sum_{k=1}^{n_2-1} \sin jr\psi_1 \sin ks\psi_2 w_{(j,k)}$. Noting that $\sin j(2n_1 - r)\psi_1 = -\sin jr\psi_1$ and $\sin k(2n_2 - s)\psi_2 = -\sin ks\psi_2$, in order to find the anti-symmetric mode, let

$$q_{(r,s)} = c\, e^{ir\psi_1} e^{is\psi_2}, \qquad q_{(2n_1-r,2n_2-s)} = c\, e^{i(2n_1-r)\psi_1} e^{i(2n_2-s)\psi_2},$$
$$q_{(2n_1-r,s)} = -c\, e^{i(2n_1-r)\psi_1} e^{is\psi_2}, \qquad q_{(r,2n_2-s)} = -c\, e^{ir\psi_1} e^{i(2n_2-s)\psi_2}, \tag{2.12}$$

with the other generalized displacements vanishing, in which c denotes an arbitrary real constant and r and s are fixed integers. Substituting equation (2.12) into the double U-transformation yields the mode

$$w_{(j,k)} = \sin jr\psi_1 \sin ks\psi_2 \qquad j = 1, 2, \ldots, 2n_1, \qquad k = 1, 2, \ldots, 2n_2 \tag{2.13}$$

neglecting an arbitrary constant factor. The mode is only dependent on a pair of integers r and s representing the numbers of the half waves in x- and y-directions respectively for the original network.

Let us return to consider the forced vibration. The solution for $q_{(r,s)}$ of equations (2.5) and (2.8) can be written as

$$q_{(r,s)} = \frac{f_{(r,s)}}{2K_1 + 2K_2 - M\omega^2 - 2K_1 \cos r\psi_1 - 2K_2 \cos s\psi_2}$$

$$r \neq n_1, 2n_1 \qquad \text{and} \qquad s \neq n_2, 2n_2 \tag{2.14}$$

with the other $q_{(r,s)}$ vanishing, i.e., equation (2.9b). Substituting equations (2.14) and (2.6) into (2.3a) results in

$$w_{(j,k)} = \sum_{j_1=1}^{2n_1} \sum_{k_1=1}^{2n_2} \beta^*_{(j,k)(j_1,k_1)} F_{(j_1,k_1)} \tag{2.15}$$

where

$$\beta^*_{(j,k)(j_1,k_1)} = \frac{1}{4n_1n_2} \sum_{r=1}^{2n_1-1} \sum_{s=1}^{2n_2-1} \frac{e^{i(j-j_1)r\psi_1}\, e^{i(k-k_1)s\psi_2}}{2K_1 + 2K_2 - M\omega^2 - 2K_1 \cos r\psi_1 - 2K_2 \cos s\psi_2}$$

$$j, j_1 = 1, 2, \ldots, 2n_1, \qquad k, k_1 = 1, 2, \ldots, 2n_2 \tag{2.16}$$

$\beta^*_{(j,k)(j_1,k_1)}$ is the harmonic influence coefficient for the $2n_1 \times 2n_2$ equivalent network without the post-supports. The * does not denote a complex conjugation in this paper.

2.2. GOVERNING EQUATION AND THE EXPLICIT SOLUTION

Consider now the equivalent network shown in Figure 2. At the outset, one can replace the supports by the supporting reactions. By invoking the superposition principle, the harmonic vibration equation can be written in terms of the harmonic influence coefficients as

$$w_{(j,k)} = w^0_{(j,k)} + w^*_{(j,k)} \tag{2.17}$$

and

$$w^0_{(j,k)} = \sum_{j_1=1}^{2m_1} \sum_{k_1=1}^{2m_2} \beta^*_{(j,k)(j_1p_1,k_1p_2)} F_{(j_1,k_1)} \tag{2.18a}$$

$$w^*_{(j,k)} = \sum_{j_1=1}^{2n_1} \sum_{k_1=1}^{2n_2} \beta^*_{(j,k)(j_1,k_1)} F_{(j_1,k_1)} \tag{2.18b}$$

where $P_{(j_1,k_1)}$ denotes the supporting reaction acting at the node (j_1p_1, k_1p_2); p_1, p_2 are the structural parameters as shown in Figure 1; $w^*_{(j,k)}$ and $w^0_{(j,k)}$ represents the displacements caused by the loading and the supporting reactions acting on the equivalent network without the post-supports, respectively; $\beta^*_{(j,k)(j_1,k_1)}$ has been defined as equation (2.16).

Substituting equations (2.16) and (2.7) into equation (2.18b) results in

$$w^*_{(j,k)} = \frac{4}{n_1 n_2} \sum_{r=1}^{n_1-1} \sum_{s=1}^{n_2-1} \sum_{j_1=1}^{n_1-1} \sum_{k_1=1}^{n_2-1} \times \frac{\sin jr\psi_1 \sin ks\psi_2 \sin j_1 r\psi_1 \sin k_1 s\psi_2}{2K_1 + 2K_2 - M\omega^2 - 2K_1 \cos r\psi_1 - 2K_2 \cos s\psi_2} F_{(j_1,k_1)}. \tag{2.19}$$

It can be verified that $w^*_{(j,k)}$ shown in equation (2.19) satisfies the anti-symmetric condition and boundary condition of the original network with $n_1 \times n_2$ mesh, i.e., $w^*_{(2n_1-j,2n_2-k)} = w^*_{(j,k)}$; $w^*_{(2n_1-j,k)} = w^*_{(j,2n_2-k)} = -w^*_{(j,k)}$, $j = 1, 2, \ldots, n_1$, $k = 1, 2, \ldots, n_2$ and $w^*_{(j,k)} = 0$, $j = 0$, n_1 or $k = 0$, n_2.

The displacements at supported nodes must be equal to zero, i.e.,

$$w^0_{(jp_1,kp_2)} + w^*_{(jp_1,kp_2)} = 0 \qquad j = 1, 2, \ldots, 2m_1, \quad k = 1, 2, \ldots, 2m_2. \tag{2.20}$$

Introducing the notation

$$W^*_{(j,k)} \equiv w^*_{(jp_1,kp_2)} \tag{2.21}$$

into the restrained condition (2.20) yields

$$\sum_{j_1=1}^{2m_1} \sum_{k_1=1}^{2m_2} \beta_{(j,k)(j_1,k_1)} P_{(j_1,k_1)} = -W^*_{(j,k)} \qquad j = 1, 2, \ldots, 2m_1, \quad k = 1, 2, \ldots, 2m_2 \tag{2.22}$$

where

$$\beta_{(j,k)(j_1,k_1)} \equiv \beta^*_{(jp_1,kp_2)(j_1p_1,k_1p_2)} = \frac{1}{4n_1 n_2} \sum_{r=1}^{2n_1-1} \sum_{s=1}^{2n_2-1} \frac{e^{i(j-j_1)r\varphi_1} e^{i(k-k_1)s\varphi_2}}{2K_1 + 2K_2 - M\omega^2 - 2K_1 \cos r\psi_1 - 2K_2 \cos s\psi_2} \tag{2.23}$$

with

$$\varphi_1 = \pi/m_1, \qquad \varphi_2 = \pi/m_2 \tag{2.24}$$

$W^*_{(j,k)}$ has been determined in equations (2.21) and (2.19).

It can be shown that the coefficients $\beta_{(j,k)(j_1,k_1)}$ of the simultaneous equations (2.22) have the cyclic periodicity, i.e.,

$$\beta_{(j,k)(1,k_1)} = \beta_{(j+1,k)(2,k_1)} = \cdots = \beta_{(j-1,k)(2m_1,k_1)}$$
$$j = 1, 2, \ldots, 2m_1; \qquad k, k_1 = 1, 2, \ldots, 2m_2 \tag{2.25a}$$

and

$$\beta_{(j,k)(j_1,1)} = \beta_{(j,k+1)(j_1,2)} = \cdots = \beta_{(j,k-1)(j_1,2m_2)}$$
$$j, j_1 = 1, 2, \ldots, 2m_1; \qquad k = 1, 2, \ldots, 2m_2. \tag{2.25b}$$

Therefore, the independent coefficients are $\beta_{(j,k)(1,1)}$ ($j = 1, 2, \ldots, 2m_1$, $k = 1, 2, \ldots, 2m_2$). The simultaneous equation (2.22) can be uncoupled by applying the double U-transformation.

Letting

$$P_{(j,k)} = \frac{1}{\sqrt{2m_1}\sqrt{2m_2}} \sum_{r=1}^{2m_1} \sum_{s=1}^{2m_2} \mathrm{e}^{\mathrm{i}(j-1)r\varphi_1}\, \mathrm{e}^{\mathrm{i}(k-1)s\varphi_2} p_{(r,s)} \tag{2.26a}$$

with its inverse transformation

$$p_{(r,s)} = \frac{1}{\sqrt{2m_1}\sqrt{2m_2}} \sum_{j=1}^{2m_1} \sum_{k=1}^{2m_2} \mathrm{e}^{-\mathrm{i}(j-1)r\varphi_1}\, \mathrm{e}^{-\mathrm{i}(k-1)s\varphi_2} P_{(j,k)} \tag{2.26b}$$

in which $\varphi_1 = \pi/m_1$ and $\varphi_2 = \pi/m_2$, and noting equations (2.25a, b), equation (2.22) can be expressed in terms of the generalized supporting reaction $p_{(r,s)}$ as

$$\left[\sum_{u=1}^{2m_1} \sum_{v=1}^{2m_2} \mathrm{e}^{-\mathrm{i}(u-1)r\varphi_1}\, \mathrm{e}^{-\mathrm{i}(v-1)s\varphi_2} \beta_{(u,v)(1,1)}\right] p_{(r,s)} = -Q^*_{(r,s)} \tag{2.27}$$

$$Q^*_{(r,s)} = \frac{1}{\sqrt{2m_1}\sqrt{2m_2}} \sum_{j=1}^{2m_1} \sum_{k=1}^{2m_2} \mathrm{e}^{-\mathrm{i}(j-1)r\varphi_1}\, \mathrm{e}^{-\mathrm{i}(k-1)s\varphi_2} W^*_{(j,k)} \tag{2.28}$$

$$\beta_{(u,v)(1,1)} = \frac{1}{4n_1 n_2} \sum_{r=1}^{2n_1-1} \sum_{s=1}^{2n_2-1} \frac{\mathrm{e}^{\mathrm{i}(u-1)r\varphi_1}\, \mathrm{e}^{\mathrm{i}(v-1)s\varphi_2}}{2K_1 + 2K_2 - M\omega^2 - 2K_1 \cos r\psi_1 - 2K_2 \cos s\psi_2}. \tag{2.29}$$

Substituting equation (2.29) into equation (2.27) results in

$$p_{(r,s)} = -\frac{Q^*_{(r,s)}}{A_{(r,s)}} \qquad r = 1, 2, \ldots, 2m_1; \quad s = 1, 2, \ldots, 2m_2 \tag{2.30}$$

where

$$A_{(r,s)} = \frac{1}{p_1 p_2} \sum_{j=1}^{p_1} \sum_{k=1}^{p_2} \{2K_1 + 2K_2 - M\omega^2 - 2K_1 \cos[r + (j-1)2m_1]\psi_1$$
$$-2K_2 \cos[s + (k-1)2m_2]\psi_2\}^{-1}. \tag{2.31}$$

If the specific load and structural parameters are given, the amplitudes of the nodal displacement and supporting reaction can be calculated from the above equations.

Consider now the natural vibration, i.e., $F_{(j,k)}$, $w^*_{(j,k)}$ and $Q^*_{(r,s)}$ are equal to zero, equation (2.30) becomes

$$p_{(r,s)} A_{(r,s)} = 0 \qquad r = 1, 2, \ldots, 2m_1; \qquad s = 1, 2, \ldots, 2m_2. \tag{2.32}$$

It can be proved that

$$p_{(r,s)} \equiv 0 \qquad r = m_1, 2m_1 \qquad \text{or} \qquad s = m_2, 2m_2.$$

When the generalized supporting reactions $p_{(r,s)}$ are not identically equal to zero, the independent frequency equation is

$$A_{(r,s)} = 0 \qquad r = 1, 2, \ldots, m_1 - 1; \qquad s = 1, 2, \ldots, m_2 - 1 \qquad (2.33)$$

and $A_{(r,s)}$ is the function of ω as shown in equation (2.31).

Consider now the natural mode. From the definition of $A_{(r,s)}$ shown in equation (2.31), it can be verified that $A_{(r,s)} \equiv A_{(2m_1 - r,s)} \equiv A_{(r,2m_2 - s)} \equiv A_{(2m_1 - r,2m_2 - s)}$. Corresponding to each natural frequency satisfying $A_{(r,s)} = 0$, four generalized supporting reactions $p_{(r,s)}$, $p_{(2m_1 - r,s)}$, $p_{(r,2m_2 - s)}$, $p_{(2m_1 - r,2m_2 - s)}$ can be equal to different constants for the extended network. We need to find the anti-symmetric mode. We must let

$$p_{(r,s)} = c\, e^{ir\varphi_1} e^{is\varphi_2}, \qquad p_{(2m_1 - r,2m_2 - s)} = c\, e^{i(2m_1 - r)\varphi_1} e^{i(2m_2 - s)\varphi_2},$$

$$p_{(2m_1 - r,s)} = -c\, e^{i(2m_1 - r)\varphi_1} e^{is\varphi_2}, \qquad p_{(r,2m_2 - s)} = -c\, e^{ir\varphi_1} e^{i(2m_2 - s)\varphi_2}, \qquad (2.34)$$

with the other $p_{(j,k)}$ vanishing, in which c denotes an arbitrary real constant.

Substituting equation (2.34) into equation (2.26a) yields

$$P_{(j,k)} = c \sin jr\varphi_1 \sin ks\varphi_2 \qquad j = 1, 2, \ldots, 2m_1, \qquad k = 1, 2, \ldots, 2m_2. \quad (2.35)$$

The corresponding mode can be found by substituting equation (2.35) and the values of r, s and ω into equation (2.18a).

When the supporting reactions are identically equal to zero, i.e., all of the supported nodes lie in the nodal lines of the mode for the network without the post-supports, the frequency equation should be expressed as equation (2.10a) in which $r = m_1, 2m_1, \ldots, (p_1 - 1)m_1$ or $s = m_2, 2m_2, \ldots, (p_2 - 1)m_2$.

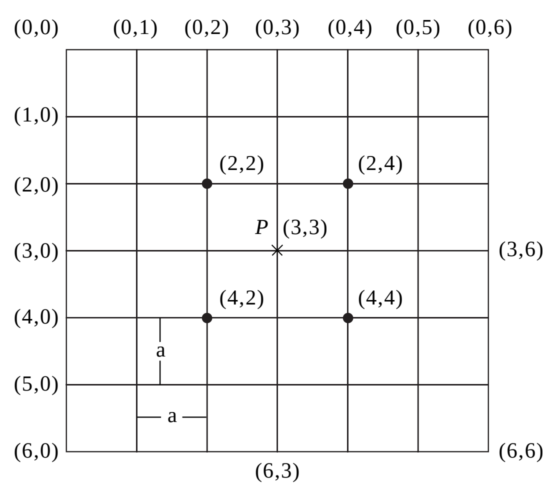

Figure 3. 6 × 6 network with 2 × 2 supports.

TABLE 1

Natural frequency (1)

(r, s)	(1, 1)	(1, 2)	(2, 1)	(2, 2)
ω^2	4 $4-\sqrt{6}$ $4+\sqrt{6}$	4 2 6	4 2 6	4 $4-\sqrt{2}$ $4+\sqrt{2}$
Multiplier	K/M			

3. EXAMPLE

Consider a uniform square network with 6×6 mesh and 2×2 internal supports as shown in Figure 3. The specific structural parameters can be written down as

$$n_1 = n_2 = 6, \qquad m_1 = m_2 = 3, \qquad p_1 = p_2 = 2, \qquad \psi_1 = \psi_2 = \frac{\pi}{6}, \qquad \varphi_1 = \varphi_2 = \frac{\pi}{3};$$

$$a = b, \qquad T_1 = T_2, \qquad K_1 = K_2 \equiv K. \tag{3.1}$$

3.1. NATURAL VIBRATION

When the supporting reactions are not identically equal to zero, the frequency equation should be equation (2.33). Substituting equations (3.1) and (2.31) into equation (2.33) gives

$$\frac{1}{4}\sum_{j=1}^{2}\sum_{k=1}^{2}\left\{4K - M\omega^2 - 2K\cos\,[r+6(j-1)]\,\frac{\pi}{6} - 2K\cos\,[s+6(k-1)]\,\frac{\pi}{6}\right\}^{-1} = 0$$

$$r = 1, 2; \qquad s = 1, 2. \tag{3.2}$$

The roots for ω^2 of the above frequency equation are summarized in Table 1.

When the supporting reactions are identically equal to zero, the corresponding frequency equation can be obtained from equation (2.10a), where one of r and s must be equal to 3 for the present case, i.e.,

$$4K - M\omega^2 - 2K\cos r\,\frac{\pi}{6} - 2K\cos s\,\frac{\pi}{6} = 0$$

$$r = 3, s = 1, 2, \ldots, 5 \qquad \text{and} \qquad s = 3, r = 1, 2, \ldots, 5 \tag{3.3}$$

because r and s on the left-hand side of equation (3.3) are in agreement with the numbers of the half wave in x- and y-directions for the original network. When r or s is equal to 3, all of the supported nodes must necessarily lie on the nodal

TABLE 2

Natural frequency (2)

(r, s)	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(1, 3)	(2, 3)	(4, 3)	(5, 3)
ω^2	$4-\sqrt{3}$	3	4	5	$4+\sqrt{3}$	$4-\sqrt{3}$	3	5	$4+\sqrt{3}$
Multiplier					K/M				

lines of the corresponding mode, i.e., the supporting reactions are identically equal to zero. The corresponding frequencies can be expressed as

$$\omega^2 = \frac{K}{M}\left(4 - 2\cos r\frac{\pi}{6} - 2\cos s\frac{\pi}{6}\right)$$

$$r = 3, s = 1, 2, \ldots, 5 \quad \text{and} \quad s = 3, r = 1, 2, \ldots, 5. \tag{3.4}$$

The result is summarized in Table 2.

The total number of natural frequencies is equal to 21 which is in agreement with the number of freedoms of the original network.

Next, consider the natural modes corresponding to the supporting reactions that do not vanish. The mode can be obtained by inserting the values of r, s, ω and the supporting reactions shown in equation (2.35) into equation (2.18a). Consider now the basic mode corresponding to the lowest natural frequency. The parameter r, s, ω can be obtained from Table 1 as

$$r = 1, \qquad s = 1 \tag{3.5a}$$

TABLE 3

Natural mode $w_{(j,k)}$ *corresponding to* $\omega^2 = (4-\sqrt{6})(K/M)$

j \ k	0	1	2	3	4	5	6	$P_{(j,k)}$
0	0	0	0	0	0	0	0	
1	0	$\frac{1}{4}$	$\frac{\sqrt{6}}{8}$	$\frac{1}{2}$	$\frac{\sqrt{6}}{8}$	$\frac{1}{4}$	0	
2	0	$\frac{\sqrt{6}}{8}$	0	$\frac{\sqrt{6}}{4}$	0	$\frac{\sqrt{6}}{8}$	0	$P_{(j,k)} = -\frac{3\sqrt{6}}{4}K$
3	0	$\frac{1}{2}$	$\frac{\sqrt{6}}{4}$	1	$\frac{\sqrt{6}}{4}$	$\frac{1}{2}$	0	$j, k = 1, 2$
4	0	$\frac{\sqrt{6}}{8}$	0	$\frac{\sqrt{6}}{4}$	0	$\frac{\sqrt{6}}{8}$	0	
5	0	$\frac{1}{4}$	$\frac{\sqrt{6}}{8}$	$\frac{1}{2}$	$\frac{\sqrt{6}}{8}$	$\frac{1}{4}$	0	
6	0	0	0	0	0	0	0	

TABLE 4

Natural mode 1, $w_{(j,k)}$ corresponding to $\omega^2 = 4K/M$ and $(r, s) = (1, 1)$

j	k	0	1	2	3	4	5	6	$P_{(j,k)}$
0		0	0	0	0	0	0	0	
1		0	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$P_{(1,1)} = P_{(1,2)}$
2		0	$\frac{1}{2}$	0	1	0	$\frac{1}{2}$	0	$= P_{(2,1)} = P_{(2,2)} = 0$*
3		0	0	-1	0	-1	0	0	
4		0	$\frac{1}{2}$	0	1	0	$\frac{1}{2}$	0	
5		0	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	
6		0	0	0	0	0	0	0	

* If $K_1 \neq K_2$, the supporting reactions are non-zero.

and

$$\omega^2 = (4 - \sqrt{6})\,\frac{K}{M}. \tag{3.5b}$$

Substituting equations (3.1) and (3.5a, b) into equation (2.35), gives

$$P_{(j,k)} = \tfrac{3}{4}c, \qquad P_{(6-j,6-k)} = P_{(j,k)}, \qquad P_{(6-j,k)} = P_{(j,6-k)} = -P_{(j,k)} \qquad j, k = 1, 2 \tag{3.6}$$

while the other $P_{(j,k)}$ vanished. Inserting equations (2.16), (3.1), (3.5b) and (3.6) into equation (2.18a), the mode can be found as shown in Table 3, in which the mode has been normalized according to the maximum amplitude being 1.

Obviously, the mode shown in Table 3 satisfies the boundary condition and restrained condition at supported nodes. It can be verified that the free vibration equation shown in equation (2.1) with $F_{(j,k)}$ vanishing is satisfied at the free nodes.

TABLE 5

Natural mode 2, $w_{(j,k)}$ corresponding to $\omega^2 = 4K/M$ and $(r, s) = (1, 2)$

j	k	0	1	2	3	4	5	6	$P_{(j,k)}$
0		0	0	0	0	0	0	0	
1		0	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$P_{(1,1)} = P_{(2,1)} = K$
2		0	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	0	$P_{(1,2)} = P_{(2,2)} = -K$
3		0	0	-1	0	1	0	0	
4		0	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	0	
5		0	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	
6		0	0	0	0	0	0	0	

TABLE 6

Natural mode 3, $w_{(j,k)}$ corresponding to $\omega^2 = 4K/M$ and $(r, s) = (2, 1)$

j k	0	1	2	3	4	5	6	$P_{(j,k)}$
0	0	0	0	0	0	0	0	
1	0	0	1/2	0	1/2	0	0	$P_{(1,1)} = P_{(1,2)} = K$
2	0	−1/2	0	−1	0	−1/2	0	$P_{(2,1)} = P_{(2,2)} = -K$
3	0	0	0	0	0	0	0	
4	0	1/2	0	1	0	1/2	0	
5	0	0	−1/2	0	−1/2	0	0	
6	0	0	0	0	0	0	0	

Noting Tables 1 and 2, it is interesting that there are five independent modes corresponding to one natural frequency, i.e., $\omega^2 = 4K/M$, where four modes are corresponding to $P_{(j,k)} \neq 0$. They can be found by using the same procedure described in the above. The results are shown in Tables 4–7.

The other mode where all the supported nodes lie in its nodal lines can be obtained by substituting equations (3.1) and $r = s = 3$ into equation (2.13) as

$$w_{(j,k)} = \sin j\frac{\pi}{2}\sin k\frac{\pi}{2}. \tag{3.7}$$

The results are as shown in Table 8.

In the same way as the above, the other modes can also be found without any difficulty.

TABLE 7

Natural mode 4, $w_{(j,k)}$ corresponding to $\omega^2 = 4K/M$ and $(r, s) = (2, 2)$

j k	0	1	2	3	4	5	6	$P_{(j,k)}$
0	0	0	0	0	0	0	0	
1	0	0	−1	0	1	0	0	$P_{(1,1)} = P_{(2,2)} = 0$*
2	0	1	0	0	0	−1	0	$P_{(2,1)} = P_{(1,2)} = -P_{(1,1)}$
3	0	0	0	0	0	0	0	
4	0	−1	0	0	0	1	0	
5	0	0	1	0	−1	0	0	
6	0	0	0	0	0	0	0	

* If $K_1 \neq K_2$, the supporting reactions are non-zero.

TABLE 8

Natural mode 5, $w_{(j,k)}$ corresponding to $\omega^2 = 4K/M$

j \ k	0	1	2	3	4	5	6	$P_{(j,k)}$
0	0	0	0	0	0	0	0	
1	0	1	0	−1	0	1	0	$P_{(1,1)} = P_{(1,2)}$
2	0	0	0	0	0	0	0	$= P_{(2,1)} = P_{(2,2)} \equiv 0$
3	0	−1	0	1	0	−1	0	
4	0	0	0	0	0	0	0	
5	0	1	0	−1	0	1	0	
6	0	0	0	0	0	0	0	

3.2. FORCED VIBRATION

Consider the same network shown in Figure 3 subjected to the harmonic loading $P\,e^{i\omega t}$ acting at its center, i.e.,

$$F_{(3,3)} = F_{(9,9)} = P, \qquad F_{(3,9)} = F_{(9,3)} = -P$$
$$F_{(j,k)} = 0 \qquad j \neq 3, 9 \qquad \text{or} \qquad k \neq 3, 9. \tag{3.8}$$

Noting the definition of $W^*_{(j,k)}$ shown in equation (2.21), substituting equations (3.1), (3,8) and (2.16) into equation (2.18b) results in

$$W^*_{(1,1)} = W^*_{(1,2)} = W^*_{(2,1)} = W^*_{(2,2)} = \frac{2P}{K(4-\Omega)(4-8\Omega+\Omega^2)}$$
$$W^*_{(6-j,6-k)} = W^*_{(j,k)}, \qquad W^*_{(6-j,k)} = W^*_{(j,6-k)} = -W^*_{(j,k)} \qquad j, k = 1, 2.$$
$$W^*_{(j,k)} = 0 \qquad j = 3, 6 \qquad \text{or} \qquad k = 3, 6 \tag{3.9}$$

where Ω denotes the non-dimensional parameter of frequency as shown in the following equation

$$\Omega = \frac{M\omega^2}{K}. \tag{3.10}$$

Substituting equations (3.1) and (3.9) into equation (2.28) yields

$$Q^*_{(1,1)} = (1 - i\sqrt{3})W^*_{(1,1)}, \qquad Q^*_{(5,5)} = (1 + i\sqrt{3})W^*_{(1,1)}$$
$$Q^*_{(1,5)} = Q^*_{(5,1)} = 2W^*_{(1,1)} \tag{3.11}$$

while the other $Q^*_{(r,s)}$ vanished.

Inserting equation (3.1) into equation (2.31), gives

$$A_{(1,1)} = A_{(1,5)} = A_{(5,1)} = A_{(5,5)} = \frac{1}{K}\frac{10-8\Omega+\Omega^2}{(4-\Omega)(4-8\Omega+\Omega^2)}. \tag{3.12}$$

Substituting equations (3.11) and (3.12) into equation (2.30) yields

$$p_{(1,1)} = -\frac{2(1 - i\sqrt{3})P}{10 - 8\Omega + \Omega^2}, \qquad p_{(5,5)} = -\frac{2(1 + i\sqrt{3})P}{10 - 8\Omega + \Omega^2},$$

$$p_{(1,5)} = p_{(5,1)} = -\frac{4P}{10 - 8\Omega + \Omega^2}, \qquad p_{(r,s)} = 0, \qquad r \neq 1, 5 \qquad \text{or} \qquad s \neq 1, 5. \tag{3.13}$$

Substituting equations (3.1) and (3.13) into equation (2.26a), the amplitudes of supporting reaction can be obtained as

$$P_{(1,1)} = P_{(1,2)} = P_{(2,1)} = P_{(2,2)} = -\frac{2P}{10 - 8\Omega + \Omega^2}$$

$$P_{(6-j,6-k)} = P_{(j,k)}, \qquad P_{(j,6-k)} = P_{(6-j,k)} = -P_{(j,k)} \qquad j, k = 1, 2$$

$$P_{(j,k)} = 0 \qquad j = 3, 6 \qquad \text{or} \qquad k = 3, 6. \tag{3.14}$$

All of the nodal displacements can be found by substituting equations (3.1), (3.8), (3.14) and (2.16) into equations (2.18a, b) and (2.17). Consider now the displacement of the loaded node, i.e., $w_{(3,3)}$. The final result can be expressed as

$$w_{(3,3)} = H(\Omega)\frac{P}{K} \tag{3.15}$$

where

$$H(\Omega) = \frac{(\Omega^2 - 8\Omega)^2 + 27(\Omega^2 - 8\Omega) + 178}{(4 - \Omega)(10 - 8\Omega + \Omega^2)(13 - 8\Omega + \Omega^2)} \tag{3.16}$$

where Ω has been defined as shown in equation (3.10).

From equation (3.16), it can be shown that, when $H(\Omega)$ approaches a finite value at a resonance frequency, the force is acting at a nodal point/line of the corresponding mode.

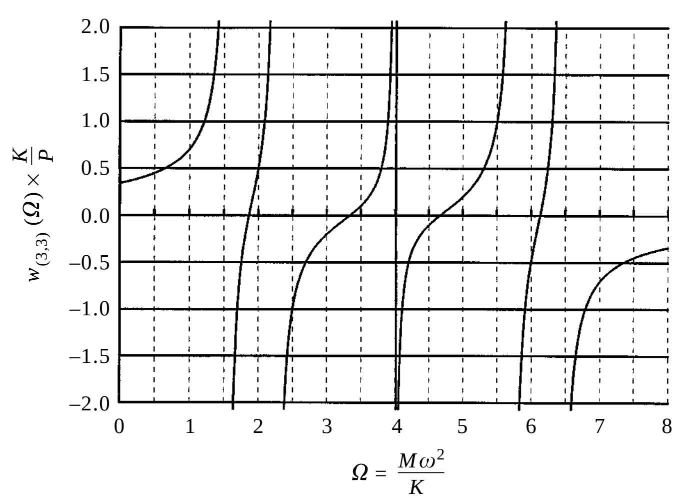

Figure 4. Frequency response curve, $w_{(3,3)}$ versus Ω.

The frequency response curve governed by equations (3.15) and (3.16), $w_{(3,3)}$ versus Ω is plotted in Figure 4.

4. CONCLUSION

In the present work, the application of the double U-transformation has been extended to dynamic analysis of rectangular cable networks with periodic supports in x- and y-directions. The structures considered belong to the category of bi-periodic structures. In order to fully utilize the periodicity property, the method presented in this paper requires the application of the double U-transformation two times altogether. At first, by applying the double U-transformation to the harmonic vibration equation, the harmonic influence coefficients are found. Then by applying the double U-transformation to the governing equation in terms of the harmonic influence coefficients, it is uncoupled into a set of single degree of freedom equations and that leads to the exact solution.

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